7.

IDENTITIES

Reciprocal Identities

$\cos\alpha = \dfrac{1}{\sec\alpha} \iff \sec\alpha = \dfrac{1}{\cos\alpha}$

$\sin\alpha = \dfrac{1}{\csc\alpha} \iff \csc\alpha = \dfrac{1}{\sin\alpha}$

$\tan\alpha = \dfrac{1}{\cot\alpha} \iff \cot\alpha = \dfrac{1}{\tan\alpha}$

Quotient Identities

$\tan\beta = \dfrac{\sin\beta}{\cos\beta} \qquad \cot\beta = \dfrac{\cos\beta}{\sin\beta}$

Pythagorean Identities

$\cos^2\gamma + \sin^2\gamma = 1 \iff \sin^2\gamma = 1 - \cos^2\gamma$

$\tan^2\gamma + 1 = \sec^2\gamma \iff \tan^2\gamma = \sec^2\gamma - 1$

$\cot^2\gamma + 1 = \csc^2\gamma \iff \cot^2\gamma = \csc^2\gamma - 1$

Cosine Sum & Difference Identities

$\cos(x - y) = \cos x\cos y + \sin x\sin y$

$\cos(x + y) = \cos x\cos y - \sin x\sin y$

Sine Sum & Difference Identities

$\sin(x - y) = \sin x\cos y - \cos x\sin y$

$\sin(x + y) = \sin x\cos y + \cos x\sin y$

Double Angle Identities

$$\begin{aligned}\cos 2\lambda &= \cos^2\lambda - \sin^2\lambda\\ &= 2\cos^2\lambda - 1\\ &= 1 - 2\sin^2\lambda\end{aligned}$$

$\sin 2\lambda = 2\sin\lambda\cos\lambda$

$\tan 2\lambda = \dfrac{2\tan\lambda}{1-\tan^2\lambda}$

Half-Angle Identities

$\cos\frac{1}{2}\mu = \pm\sqrt{\dfrac{1+\cos\mu}{2}}$

$\sin\frac{1}{2}\mu = \pm\sqrt{\dfrac{1-\cos\mu}{2}}$

$\tan\frac{1}{2}\mu = \pm\sqrt{\dfrac{1-\cos\mu}{1+\cos\mu}}$

Negative Identities

$\sin(-\zeta) = -\sin\zeta \iff \csc(-\zeta) = -\csc\zeta$

$\cos(-\zeta) = \cos\zeta \iff \sec(-\zeta) = \sec\zeta$

$\tan(-\zeta) = -\tan\zeta \iff \cot(-\zeta) = -\cot\zeta$

Prove:

$\dfrac{1-\tan^2\partial}{1+\tan^2\partial} \stackrel{?}{=} \cos 2\partial$

(1) (tan²∂)

$\dfrac{1-\tan^2\partial}{1+\tan^2\partial} = \dfrac{\left(\dfrac{\cos^2\partial}{\cos^2\partial} - \dfrac{\sin^2\partial}{\cos^2\partial}\right)}{\sec^2\partial}$

$\left(\dfrac{1}{\sec^2\partial}\right)$

$= \left(\dfrac{\cos^2\partial}{\cancel{\cos^2\partial}} - \dfrac{\sin^2\partial}{\cancel{\cos^2\partial}}\right)\cancel{\cos^2\partial}$

$= \cos^2\partial - \sin^2\partial = \cos 2\partial$ (Identity)

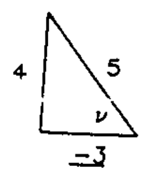
Evaluate $\cos 3\lambda$ in terms of $\cos\lambda$:

$$\begin{aligned}\cos 3\lambda &= \cos(2\lambda+\lambda)\\ &= \cos 2\lambda\cos\lambda - \sin 2\lambda\sin\lambda\\ &= (2\cos^2\lambda - 1)\cos\lambda - (2\sin\lambda\cos\lambda)\sin\lambda\\ &= (2\cos^2\lambda - 1)\cos\lambda - (2\sin^2\lambda\cos\lambda)\\ &= (2\cos^2\lambda - 1)\cos\lambda - 2(1-\cos^2\lambda)\cos\lambda\\ &= 2\cos^3\lambda - \cos\lambda - 2\cos\lambda + 2\cos^3\lambda\\ &= \underline{\underline{4\cos^3\lambda - 3\cos\lambda}}\end{aligned}$$

Means ν is in Quadrant 2.

$\sin\nu = \frac{4}{5}$ and $\cos\nu < 0$

Find $\tan\frac{1}{2}\nu$.



$\cos\nu = \frac{-3}{5}$

$\tan\frac{1}{2}\nu = \pm\sqrt{\dfrac{1-\cos\nu}{1+\cos\nu}}$

Tangent is negative in Quadrant 2.

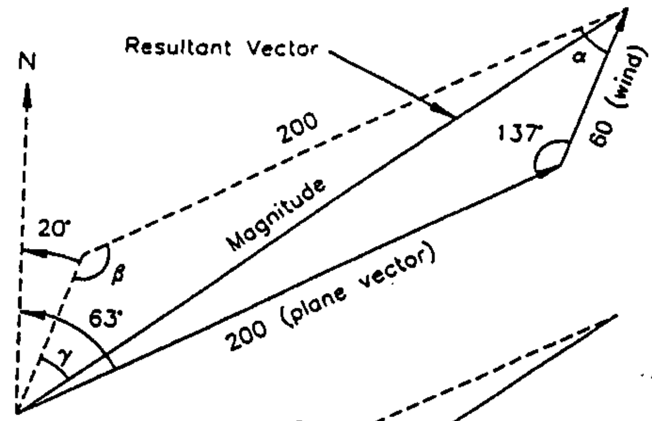
$\tan\frac{1}{2}\nu = -\sqrt{\dfrac{1-\frac{-3}{5}}{1+\frac{-3}{5}}} = -\sqrt{\dfrac{\frac{8}{5}}{\frac{2}{5}}} = -\sqrt{\frac{8}{5}\left(\frac{5}{2}\right)} = \underline{\underline{-2}}$

VECTORS

Most word problems will be solved by making a parallelogram out of the vectors involved, then using the Law of Cosines to find the resultant vector.

Example:

A plane flies at airspeed of 200mph and a bearing of 63°.
A wind is blowing 60mph at a bearing of 20°.
What is the actual speed and bearing of the plane?

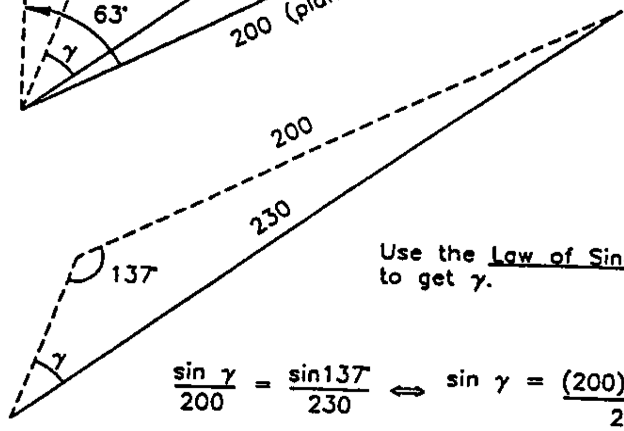


Solution:

The acute angle of the parallelogram (α) is 43° (63°−20°).
The obtuse angle of the parallelogram (β) is 137° (180°−43°).

Use the Law of Cosines to get magnitude.

$$c^2 = (200)^2 + (60)^2 - 2(200)(60)\cos 137°$$
$$c^2 = (40000) + (3600) - 24000(-0.8888)$$
$$c^2 = 43500$$
$$c = \underline{230\text{mph}}$$



Use the Law of Sines to get γ.

$$\frac{\sin\gamma}{200} = \frac{\sin 137°}{230} \Leftrightarrow \sin\gamma = \frac{(200)\sin 137°}{230}$$

$$\sin\gamma = 1.8735 \Leftrightarrow \underline{\gamma = 16.8888°}$$

The actual bearing of the plane is 16.8888°(γ) + 20° = $\underline{36.8888°}$

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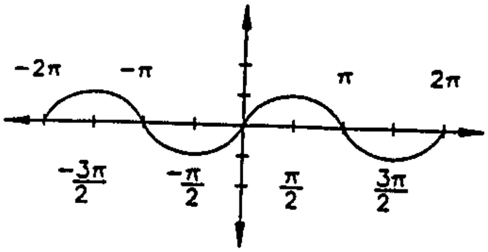
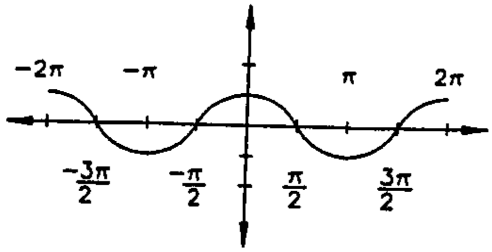
Sine Difference & Sine Sum Identities

$$\sin(x - y) = \sin x\cos y - \cos x\sin y$$
$$\sin(x + y) = \sin x\cos y + \cos x\sin y$$

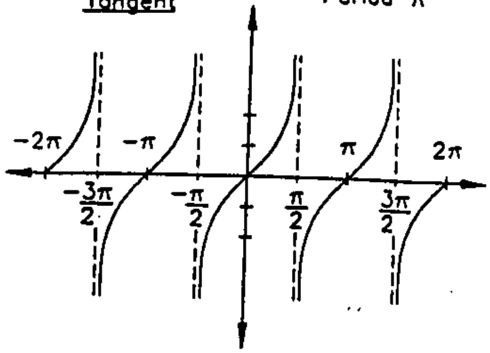
Basic Sine Period 2π



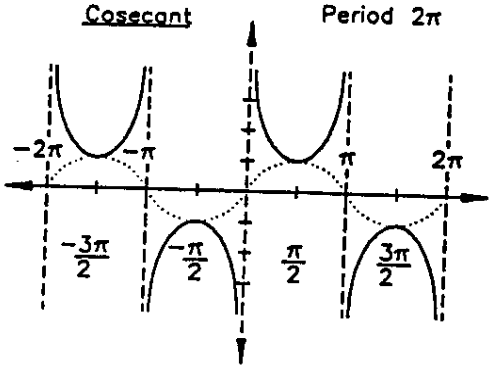
Basic Cosine Period 2π



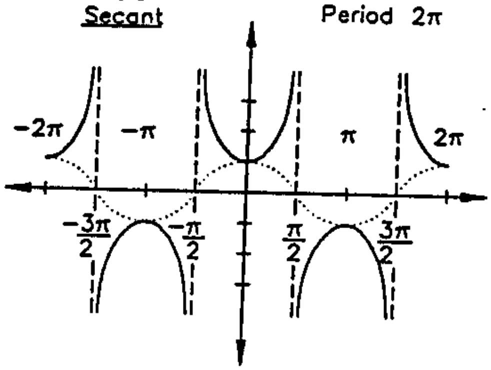
Basic Tangent Period π



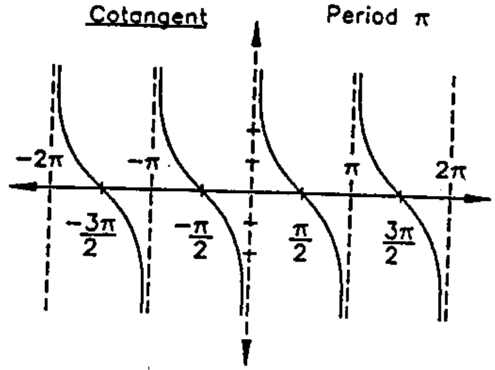
Basic Cosecant Period 2π



Basic Secant Period 2π



Basic Cotangent Period π



Amplitude → A; Angle (in radians) → x

$$f(y) = A \sin(Bx + C)$$

Frequency → B; Horizontal Shift → C

Starting Point: $(Bx + C) = 0$ (sin, cos, csc, sec & cot)

Asymptote: $(Bx + C) = \frac{\pi}{2}$ (tan)

New Period: $\frac{\text{basic period}}{\text{coefficient B}}$ ← For tan & cot, this is the distance between asymptotes.

New Interval: $\frac{\text{New Period}}{4}$

Amplitude: $|A|$

Example:

Since it's secant, first make a cosine wave

$f(t) = {}^{-}3 \sec(-\frac{t}{2} + \frac{\pi}{3})$

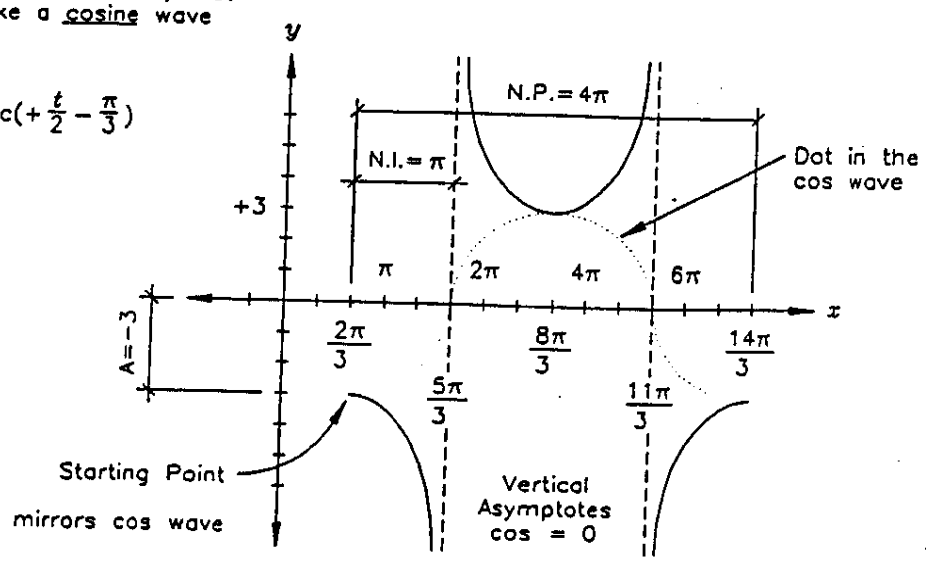
$f(t) = {}^{-}3 \sec(-\frac{t}{2} + \frac{\pi}{3}) \Leftrightarrow f(t) = -3 \sec(+\frac{t}{2} - \frac{\pi}{3})$

Starting Point: $\frac{t}{2} - \frac{\pi}{3} = 0 \Leftrightarrow t = \frac{2\pi}{3}$

New Period: $\frac{2\pi}{\frac{1}{2}} \Leftrightarrow 4\pi$

New Interval: $\frac{4\pi}{4} \Leftrightarrow \pi$

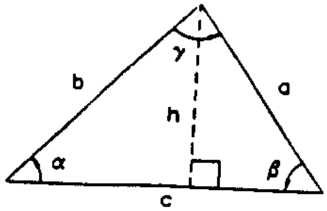
Amplitude: $|-3|$



LAW OF SINES

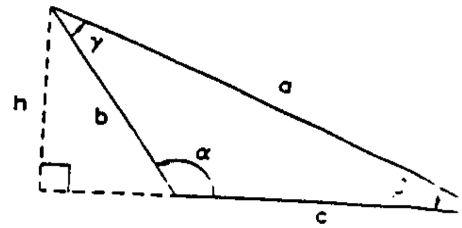
2 angles + 1 side or
2 sides + opposite angle

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$



$\sin\alpha = \frac{h}{b}$

$h = b\sin\alpha$



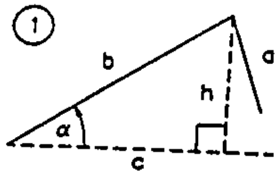
Example:
Given: 2 angles and 1 side

$\alpha = 45^\circ$
$\beta = 60^\circ$
$a = 4.6$

Since you know 2 angles, $\gamma = 75^\circ$ $(180^\circ - 45^\circ - 60^\circ)$

THE FOUR POSSIBILITIES

(1)



$a < b\sin\alpha$
No triangle

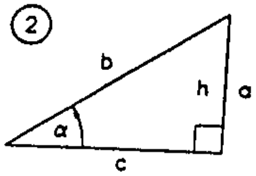
Always solve the smallest angle first with Law of Sines. Your calculator will figure the sine of an obtuse angle as if it were acute. It will mess you up!

Solution:

$$\frac{b}{\sin 60^\circ} = \frac{4.6}{\sin 45^\circ} \iff b = \frac{(4.6)\sin 60^\circ}{\sin 45^\circ} \iff b = 5.6338$$

$$\frac{c}{\sin 75^\circ} = \frac{4.6}{\sin 45^\circ} \iff c = \frac{(4.6)\sin 75^\circ}{\sin 45^\circ} \iff c = 6.2837$$

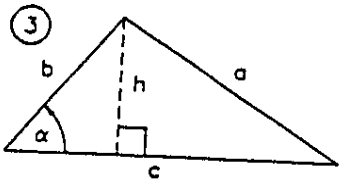
(2)



$a = b\sin\alpha$
1 right triangle

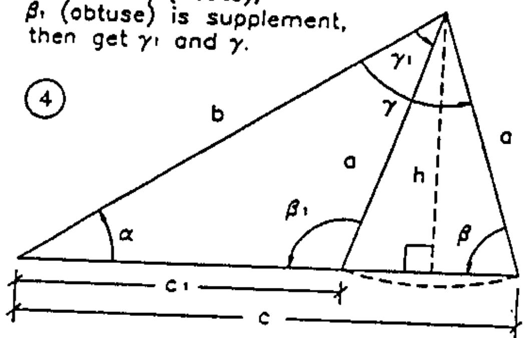
Ambiguous Case:
Solve for β (acute); β_1 (obtuse) is supplement, then get γ_1 and γ.

(3)



$a \geq b$
1 triangle

(4)



$b\sin\alpha < a < b$
2 triangles

△ #1 is a,b,c   △ #2 is a ,b,c₁

AREA OF TRIANGLE

$$K = \frac{1}{2}ab(\sin\gamma)$$

(half the product of two sides and the sine of the included angle)

LAW OF COSINES

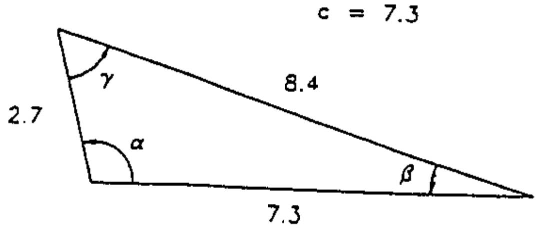
2 sides + included angle (a,b,γ) or
3 sides

$c^2 = a^2 + b^2 - 2(a)(b)\cos\gamma$

$\left(\begin{array}{l} a^2 = b^2 + c^2 - 2(b)(c)\cos\alpha \\ b^2 = a^2 + c^2 - 2(a)(c)\cos\beta \end{array}\right)$

Example:
Given: 3 sides

$a = 8.4$
$b = 2.7$
$c = 7.3$



Always solve the biggest angle first with Law of Cosines. Since your calculator has no problem getting the cosine of an obtuse angle, you must go for it!

Solution:

$a^2 = b^2 + c^2 - 2(b)(c)\cos\alpha$ [find α]

$(8.4)^2 = (2.7)^2 + (7.3)^2 - 2(2.7)(7.3)\cos\alpha$ [isolate $\cos\alpha$]

$8.4\ \boxed{x^2} - 2.7\ \boxed{x^2} - 7.3\ \boxed{x^2}$

Calculator sequence

$9.98 = -39.42\cos\alpha$

$-0.253171 = \cos\alpha$

$\boxed{\cos^{-1}}$ $\alpha = 104.6652^\circ$

Once you've found α, use Law of Sines to find β. Then, $180^\circ - \alpha - \beta = \gamma$.

A little ALGEBRA APPLIED TO TRIGONOMETRY

REVIEW EACH OF THE PROCEDURES BELOW AND APPLY IT TO THE TRIGONOMETRIC EXAMPLE.

1. $A^2 - B^2 = (A - B)(A + B)$

   Ex. $\sin^2(x) - \cos^2(x)$

2. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

   Ex. $\csc^3(x) - \cot^3(x)$

3. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

   Ex. $\sec^3(x) + \tan^3(x)$

4. $(A/B)(C/C) = AC/BC$

   Ex. $[\sin(x)/(1 - \cos(x))]\,[(1 + \cos(x))/(1 + \cos(x))]$

5. $A/B + C/D = (AD + BC)/BD$

   Ex. $\sin(x)/\cos(x) + \cos(x)/\sin(x)$

6. $A/B - C/D = (AD - BC)/BD$

   Ex. $\cos(x)/\sin(x) - \csc(x)/\sec(x)$

7. $A^2 - 2A + 1 = (A - 1)^2$

   Ex. $\sin^2(x) - 2\sin(x) + 1$

8. $B^2 + 2B + 1 = (B + 1)^2$

   Ex. $\cos^2(x) + 2\cos(x) + 1$

9. $2X^2 - 3X + 1 = (2x - 1)(x - 1)$

   $2\tan^2(x) - 3\tan(x) + 1$